Définition 2.7

- Les valeurs ——— k_1 et k_2 s'appellent les courbures principales de S au point P
- Les vecteurs propres w_1 et w_2 sont les directions principals de S au point P.
- $K = \det(W) = k_1 \cdot k_2$ s'appelle la courbure de Gauss
- $H = \frac{1}{2} \operatorname{trace}(W) = \frac{1}{2}(k_1 + k_2)$, s'appelle la courbure moyenne

Proposition 2.5

- Les coefficients de la matrice de Weingarten est donnée par
$$w_{11} = \frac{MF - LG}{EG - F^2}, \quad w_{21} = \frac{NF - MG}{EG - F^2}$$
$$w_{12} = \frac{LF - ME}{EG - F^2}, \quad w_{22} = \frac{MF - NE}{EG - F^2}$$
- La courbure de Gausse et la courbure moyenne
$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{-2FM + EN + LG}{2(EG - F^2)}$$

Preuve :

On a: $\begin{cases} L = \langle n, \varphi_{uu} \rangle = -\langle n_u \cdot \varphi_u \rangle \\ M = \langle n, \varphi_{uv} \rangle = -\langle n_u, \varphi_v \rangle = -\langle n_v, \varphi_u \rangle \\ N = \langle n, \varphi_{vv} \rangle = -\langle n_v, \varphi_v \rangle = \end{cases}$

$$\begin{cases} L = -\langle w_{11}\varphi_u + w_{21}\varphi_v, \varphi_u \rangle = -(w_{11}E + w_{21}F) \\ M = -\langle w_{11}\varphi_u + w_{21}\varphi_v, \varphi_v \rangle = -(w_{11}F + w_{21}G) \\ \quad = -\langle w_{12}\varphi_u + w_{22}\varphi_v, \varphi_u \rangle = -(w_{12}E + w_{22}F) \\ N = -\langle w_{12}\varphi_u + w_{22}\varphi_v, \varphi_v \rangle = -(w_{12}F + w_{22}G) \end{cases}$$

d'où

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ alors:}$$

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

mais $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \dfrac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \frac{-1}{EG-F^2}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

on obtient: $w_{11} = \dfrac{MF-LG}{EG-F^2}$, $\quad w_{12} = \dfrac{LF-ME}{EG-F^2}$

$$W_{21} = \frac{NF-MG}{EG-F^2} \quad , \quad W_{22} = \frac{MF-NE}{EG-F^2}$$

2) On $K = \det(W) = W_{11} \cdot W_{22} - W_{21} W_{12}$

$$= \frac{(MF-LG)(MF-NE)-(NF-MG)(LF-ME)}{(EG-F^2)^2}$$

$$= \frac{M^2F^2 - \cancel{MFNE} - \cancel{LGMF} + LGNE - NF^2L + \cancel{NFME} + \cancel{MGLF} - M^2GE}{(EG-F^2)^2}$$

$$= \frac{M^2(F^2-GE)+LN(GE-F^2)}{(EG-F^2)^2} = \frac{LN-M^2}{EG-F^2}$$

On a : $H = -\frac{1}{2}\,\text{trace}(W) = -\frac{1}{2}(W_{11}+W_{22}) = -\frac{1}{2}\left(\frac{MF-LG}{EG-F^2}+\frac{MF-NE}{EG-F^2}\right)$

$$= \frac{-2MF+LG+NE}{2(EG-F^2)}$$

Exemple: 2.5

Determiner les courbure K et H pour le cylindre

Solution:

$K = 0 \ , \ H = -\frac{1}{2}$

Exemple: 2.6

Determiner les courbure K et H pour Le tore

Solution:

$\varphi :]0, 2\pi[^2 \longrightarrow \mathbb{R}^3$ $\qquad b > a$

$(\alpha, \theta) \longmapsto \varphi(\alpha, \theta) = \big((b + a\sin\theta)\cdot\cos\alpha, (b+a\sin\theta)\sin\alpha, a\cos\theta$

$\varphi_\alpha = \big(-(b+a\sin\theta)\sin\alpha, (b+a\sin\theta)\cos\alpha, 0\big)$

$\varphi_\theta = \big(a\cos\theta\cdot\cos\alpha, a\cos\theta\sin\alpha, -a\sin\theta\big)$

$\varphi_{\alpha\alpha} = \big(-(b+a\sin\theta)\cos\alpha, -(b+a\sin\theta)\sin\alpha, 0\big)$

$\varphi_{\alpha\theta} = \big(-a\cos\theta\sin\alpha, a\cos\theta\cos\alpha, 0\big)$

$\varphi_{\theta\theta} = \big(-a\sin\theta\cos\alpha, -a\sin\theta\sin\alpha, -a\cos\theta\big)$

$E = \langle \varphi_\alpha, \varphi_\alpha \rangle = (b + a\sin\theta)^2$

$F = \langle \varphi_\alpha, \varphi_\theta \rangle = 0$

$G = \langle \varphi_\theta, \varphi_\theta \rangle = a^2$

$n = \dfrac{\varphi_\alpha \wedge \varphi_\theta}{\|\varphi_\alpha \wedge \varphi_\theta\|}$, $\varphi_\alpha \wedge \varphi_\theta = (b_1, b_2, b_3)$

$b_1 = -a(b+a\sin\theta)\cos\alpha\cdot\sin\theta$

$b_2 = -a(b+a\sin\theta)\sin\alpha\sin\theta$

$b_3 = -a(b+a\sin\theta)\cos\theta$

$\|\varphi_\alpha \wedge \varphi_\theta\| = \sqrt{EG - F2} = a(b + a\sin\theta)$

d'ou $n = \big(-\cos\alpha\sin\theta, -\sin\alpha\sin\theta, -\cos\theta\big)$

$L = \langle \varphi_{\alpha\alpha}, n \rangle = (b + a\sin\theta)\sin\theta$

$$M = \langle \varphi_{\alpha\theta}, n \rangle = 0$$

$$N = \langle \varphi_{\theta\theta}, n \rangle = a$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(b + a\sin\theta)\sin\theta \times a - 0}{(b + a\sin\theta)^2 \times a^2 - 0} = \frac{\sin\theta}{a(b + a\sin\theta)}$$

$$H = \frac{-2FM + EN + GL}{2(EG - F^2)} = \frac{b + 2a\sin\theta}{2a(b + a\sin\theta)}$$