$$
\begin{aligned}
\bar{R}(X,Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\
&= \bar{\nabla}_X(\nabla_Y Z + h(Y,Z)) - \bar{\nabla}_Y(\nabla_X Z + h(X,Z)) - \nabla_{[X,Y]}Z - h([X,Y],Z) \\
&= \bar{\nabla}_X\nabla_Y Z + \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y\nabla_X Z - \bar{\nabla}_Y h(X,Z) - \nabla_{[X,Y]}Z - h([X,Y],Z) \\
&= \nabla_X\nabla_Y Z + h(X,\nabla_Y Z) + \bar{\nabla}_X h(Y,Z) - \nabla_Y\nabla_X Z - h(Y,\nabla_X Z) \\
&\quad -\bar{\nabla}_Y h(X,Z) - \nabla_{[X,Y]}Z - h([X,Y],Z) \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + h(X,\nabla_Y Z) - h(Y,\nabla_X Z) \\
&\quad -h([X,Y],Z) + \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y h(X,Z) \\
&= R(X,Y)Z + h(X,\nabla_Y Z) - h(Y,\nabla_X Z) - h([X,Y],Z) + \bar{\nabla}_X h(Y,Z) \\
&\quad -\bar{\nabla}_Y h(X,Z)
\end{aligned}
$$

■

Proposition 2.2.2 *Soient N une variété Riemannienne et M une sous variété de M et $\bar{K}$,K les tenseur de courbure Riemannienne-*CHRISTOFFEL *de N, M respectivement. Alors pour tout champs de vecteurs X, Y, Z, W de M, on a*

$$
\begin{aligned}
\bar{K}(X,Y,Z,W) &= K(X,Y,Z,W) + \bar{g}(h(X,Z), h(Y,W)) \\
&\quad -\bar{g}(h(Y,Z), h(X,W))
\end{aligned}
$$

Preuve. Soient $\zeta_1, \zeta_2 \ldots\ldots, \zeta_{n-m}$ des champs de vecteurs normale orthonormale de M et soit h^i la correspondence de la seconde forme fondamental, c'est à dire

$$h(X,Y) = h^i(X,Y)\,\zeta_i \tag{2.2.2}$$

$h^i : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ est un champ tenseur de type (0.2). Alors, de $(2.1.6)$ et $(2.1.7)$, on a

$$
\begin{aligned}
\bar{R}(X,Y)Z &= R(X,Y)Z + h^i(X,\nabla_Y Z)\,\zeta_i - h^i(Y,\nabla_X Z)\,\zeta_i - h^i([X,Y],Z)\,\zeta_i \\
&\quad +\bar{\nabla}_X h^i(Y,Z)\,\zeta_i - \bar{\nabla}_Y h^i(X,Z)\,\zeta_i
\end{aligned}
$$

$$
\begin{aligned}
&= R(X,Y)\,Z + h^i(X,\nabla_Y Z)\,\zeta_i - h^i(Y,\nabla_X Z)\,\zeta_i - h^i([X,Y],Z)\,\zeta_i \\
&\quad + h^i(Y,Z)\,\bar{\nabla}_X\zeta_i + X(h^i(Y,Z))\zeta_i - h^i(X,Z)\,\bar{\nabla}_Y\zeta_i - Y(h^i(X,Z))\zeta_i \\
&= R(X,Y)\,Z + h^i(X,\nabla_Y Z)\,\zeta_i - h^i(Y,\nabla_X Z)\,\zeta_i - h^i([X,Y],Z)\,\zeta_i \\
&\quad + h^i(Y,Z)\left(-\left(A_{\zeta_i}(X)\right) + \nabla^{\perp}_X\zeta_i\right) - h^i(X,Z)\left(-\left(A_{\zeta_i}(Y)\right) + \nabla^{\perp}_Y\zeta_i\right) \\
&\quad - Y(h^i(X,Z))\zeta_i + X(h^i(Y,Z))\zeta_i \\
&= R(X,Y)\,Z + h^i(X,\nabla_Y Z)\,\zeta_i - h^i(Y,\nabla_X Z)\,\zeta_i - h^i(\nabla_X Y,Z)\,\zeta_i \\
&\quad + h^i(\nabla_Y X,Z)\,\zeta_i + X(h^i(Y,Z))\zeta_i - h^i(Y,Z)\,A_{\zeta_i}(X) + h^i(Y,Z)\,\nabla^{\perp}_X\zeta_i \\
&\quad - Y(h^i(X,Z))\zeta_i + h^i(X,Z)_i\,A_{\zeta_i}(Y) - h^i(X,Z)\,\nabla^{\perp}_Y\zeta_i \\
&= R(X,Y)\,Z + ((Xh^i(Y,Z)) - h^i(\nabla_X Y,Z) - h^i(Y,\nabla_X Z))\zeta_i \\
&\quad - (Y(h^i(X,Z)) - h^i(\nabla_Y X,Z) - h^i(X,\nabla_Y Z))\zeta_i \\
&\quad - h^i(Y,Z)\,A_{\zeta_i}(X) + h^i(X,Z)_i\,A_{\zeta_i}(Y) + h^i(Y,Z)\,\nabla^{\perp}_X\zeta_i - h^i(X,Z)\,\nabla^{\perp}_Y\zeta_i \\
&= R(X,Y)\,Z + \{(\nabla_X h^i)(X,Z) - (\nabla_Y h^i)(Y,Z)\}\zeta_i - h^i(Y,Z)\,A_{\zeta_i}(X) \qquad (2.2.3) \\
&\quad + h^i(X,Z)_i\,A_{\zeta_i}(Y) + h^i(Y,Z)\,\nabla^{\perp}_X\zeta_i - h^i(X,Z)\,\nabla^{\perp}_Y\zeta_i
\end{aligned}
$$

donc

$$(\nabla_X h^i)(X,Z) = X(h^i(Y,Z)) - h^i(\nabla_X Y,Z) - h^i(Y,\nabla_X Z)$$

pour n'importe quele champ de vecteur W sur M, on a

$$
\begin{aligned}
\bar{K}(X,Y,Z,W) &= \bar{g}\left(\bar{R}(X,Y)\,Z,W\right) \\
&= \bar{g}(-h^i(Y,Z)\,A_{\zeta_i}(X) + h^i(X,Z)_i\,A_{\zeta_i}(Y),W) + K(X,Y,Z,W) \\
&= K(X,Y,Z,W) + \bar{g}(h^i(X,Z)\,A_{\zeta_i}(Y),W) - \bar{g}(h^i(Y,Z)\,A_{\zeta_i}(X),W) \\
&= K(X,Y,Z,W) + h^i(X,Z)\,g\left(A_{\zeta_i}(Y),W\right) - h^i(Y,Z)\,g\left(A_{\zeta_i}(X),W\right) \\
&= K(X,Y,Z,W) + h^i(X,Z)\,\bar{g}\left(h(Y,W),\zeta_i\right) - h^i(Y,Z)\,\bar{g}\left(h(X,W),\zeta_i\right) \\
&= K(X,Y,Z,W) + \bar{g}\left(h(Y,W),h(X,Z)\right) - \bar{g}\left(h(X,W),h(Y,Z)\right)
\end{aligned}
$$

De plus, de l'equation (2.2.3), nous voyons que la composente normal de $\bar{R}(X,Y)\,Z$ est donné par

$$\left(\bar{R}(X,Y)Z\right)^N = \{(\nabla_X h^i)(X,Z) - (\nabla_Y h^i)(Y,Z)\}\zeta_i \qquad (2.2.4)$$
$$+h^i(Y,Z)\nabla^{\perp}_X\zeta_i - h^i(X,Z)\nabla^{\perp}_Y\zeta_i$$

■

Remarque 2.2.3 *L'equation* (2.2.1) *s'appelle* ***l'equation de*** GAUSS *et l'equation* (2.2.4) *s'appelle* ***l'equation de*** CODAZZI

Si on définit la dérivé covariant, pour la seconde forme fondamentale, qui noté par $\dddot{\nabla}_X h$,

$$\begin{aligned}
(\dddot{\nabla}_X h)(Y,Z) &= \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \\
&= \nabla^{\perp}_X (h^i(Y,Z)\zeta_i) - \{h^i(\nabla_X Y,Z) + h^i(Y,\nabla_X Z)\}\zeta_i \\
&= h^i(Y,Z)\nabla^{\perp}_X(\zeta_i) + X(h^i(Y,Z))\zeta_i - \{h^i(\nabla_X Y,Z) + h^i(Y,\nabla_X Z)\}\zeta_i \\
&= \{X(h^i(Y,Z)) - h^i(\nabla_X Y,Z) - h^i(Y,\nabla_X Z)\}\zeta_i + h^i(Y,Z)\nabla^{\perp}_X(\zeta_i) \\
&= h^i(Y,Z)\nabla^{\perp}_X(\zeta_i) + (\nabla_X h^i)(Y,Z)\zeta_i \qquad (2.2.5)
\end{aligned}$$

alors de cette equation nous voyons que l'equation du codazzi peut egalement être écrite comme

$$\left(\bar{R}(X,Y)Z\right)^N = \dddot{\nabla}_X h(Y,Z) - \dddot{\nabla}_Y h(X,Z) \qquad (2.2.6)$$

De plus, si ζ *et* η *sont deux champs de vecteurs normaux de* M, *alors, on a*

$$\begin{aligned}
\bar{K}(X,Y,\zeta,\eta) &= \bar{g}\left(\bar{\nabla}_X\bar{\nabla}_Y\zeta - \bar{\nabla}_Y\bar{\nabla}_X\zeta - \bar{\nabla}_{[X,Y]}\zeta, \eta\right) \\
&= \bar{g}\left(\bar{\nabla}_X\left(-(A_\zeta(Y)) + \nabla^{\perp}_Y\zeta\right), \eta\right) - \bar{g}\left(\bar{\nabla}_Y\left((A_\zeta(X)) + \nabla^{\perp}_X\zeta\right), \eta\right) \\
&\quad + \bar{g}((A_\zeta([X,Y])) - \nabla^{\perp}_{[X,Y]}\zeta, \eta) \\
&= -\bar{g}\left(\bar{\nabla}_X A_\zeta(Y), \eta\right) + \bar{g}\left(\bar{\nabla}_X\nabla^{\perp}_Y\zeta, \eta\right) + \bar{g}\left(\bar{\nabla}_Y A_\zeta(X), \eta\right) - \bar{g}\left(\bar{\nabla}_Y\nabla^{\perp}_X\zeta, \eta\right) \\
&\quad + \bar{g}\left(A_\zeta([X,Y]), \eta\right) - \bar{g}(\nabla^{\perp}_{[X,Y]}\zeta, \eta) \\
&= -\bar{g}\left(\nabla_X A_\zeta(Y) + h(X, A_\zeta(Y)), \eta\right) + \bar{g}\left(\nabla^{\perp}_X\nabla^{\perp}_Y\zeta, \eta\right) - \bar{g}\left(A_\zeta\left(\nabla^{\perp}_Y\zeta\right), \eta\right) \\
&\quad + \bar{g}\left(\nabla_Y A_\zeta(X) + h(Y, A_\zeta(X)), \eta\right) - \bar{g}\left(\nabla^{\perp}_Y\nabla^{\perp}_X\zeta, \eta\right) + \bar{g}\left(A_\zeta\left(\nabla^{\perp}_X\zeta\right), \eta\right) \\
&\quad - \bar{g}(\nabla^{\perp}_{[X,Y]}\zeta, \eta)
\end{aligned}$$