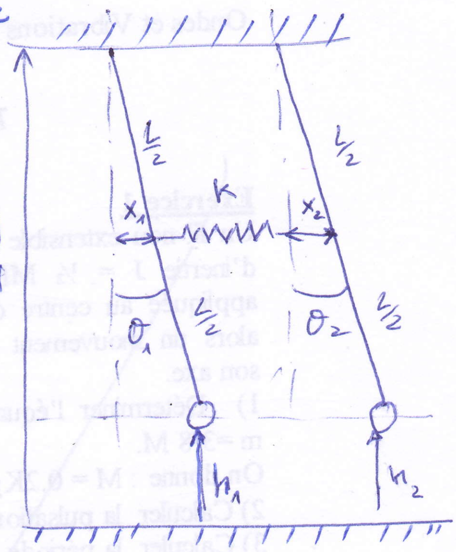
Ex 01

TD : Vibrations 2 ddl

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\theta_1}\right) - \dfrac{\partial L}{\partial \theta_1} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\theta_2}\right) - \dfrac{\partial L}{\partial \theta_2} = 0 \end{cases}$$ Eq de Lagrange



$$T = \underbrace{\frac{1}{2} m L^2 \dot\theta_1^2}_{E_c \text{ masse } 1} + \underbrace{\frac{1}{2} m L^2 \dot\theta_2^2}_{E_c \text{ masse } 2}$$

$$V = \underbrace{mg\,h_1}_{E_{PP1}} + \underbrace{mg\,h_2}_{E_{PP2}} + \underbrace{\frac{1}{2} K (x_2 - x_1)^2}_{E_{P_e} \text{ (ressort)}} \Rightarrow \begin{cases} h_1 = H - L\cos\theta_1 \\ h_2 = H - L\cos\theta_2 \end{cases}$$

$$= mgH - mgL\cos\theta_1 + mgH - mgL\cos\theta_2 + \frac{1}{2} K \Big(\underbrace{\frac{L}{2}\sin\theta_2}_{x_2} - \underbrace{\frac{L}{2}\sin\theta_1}_{x_1}\Big)^2$$

Lagrangien $L = T - V$

$$L = \frac{1}{2} m L^2 \dot\theta_1^2 + \frac{1}{2} m L^2 \dot\theta_2^2 - mgH + mg\,L\cos\theta_1 - mgH$$

$$+ mgL\cos\theta_2 - \frac{1}{2} K \left(\frac{L}{2}\sin\theta_2 - \frac{L}{2}\sin\theta_1\right)^2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = m L^2 \ddot\theta_1$$

$$\frac{\partial L}{\partial \theta_1} = -mgL\sin\theta_1 - \left(-\frac{L}{2}\cos\theta_1\right) K \left(\frac{L}{2}\sin\theta_2 - \frac{L}{2}\sin\theta_1\right)$$

$$= -mgL\sin\theta_1 + \frac{L^2}{4} K\cos\theta_1 \left(\sin\theta_2 - \sin\theta_1\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = mL^2\ddot\theta_1 + mgL\sin\theta_1 - \frac{L^2}{4} K\cos\theta_1(\sin\theta_2 - \sin\theta_1) = 0$$

petites oscillations: $\Rightarrow mL^2\ddot\theta_1 + mgL\theta_1 - \frac{L^2}{4}K\theta_2 + \frac{L^2}{4}K\theta_1 = 0$

$\cos\theta_1 = 1$, $\sin\theta_1 = \theta_1$

$$\boxed{mL^2\ddot\theta_1 + \left(mgL + K\frac{L^2}{4}\right)\theta_1 - K\frac{L^2}{4}\theta_2 = 0} \quad \ldots \boxed{\text{Eq n°1}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = mL^2\ddot\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -mgL\sin\theta_2 - \left(\frac{L}{2}\cos\theta_2\right) K \left(\frac{L}{2}\sin\theta_2 - \frac{L}{2}\sin\theta_1\right)$$

$$= -mgL\sin\theta_2 - \frac{L^2}{4}\cos\theta_2 K \left(\sin\theta_2 - \sin\theta_1\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = mL^2\ddot\theta_2 + mgL\sin\theta_2 + \frac{L^2}{4}\cos\theta_2 K(\sin\theta_2 - \sin\theta_1) = 0$$

$$mL^2\ddot\theta_2 + mgL\sin\theta_2 + \frac{L^2}{4}\cos\theta_2 K(\sin\theta_2 - \sin\theta_1) = 0$$

petites oscillations: $\Rightarrow$

$\cos\theta_2 = 1$, $\sin\theta_2 = \theta_2$ $\quad mL^2\ddot\theta_2 + mgL\theta_2 + \frac{L^2}{4}K\theta_2 - \frac{L^2}{4}\cancel{K}\,\theta_1 = 0$

$$\boxed{mL^2\ddot\theta_2 + \left(mgL + K\frac{L^2}{4}\right)\theta_2 - \frac{L^2}{4}K\theta_1 = 0} \quad \ldots \boxed{\text{Eq n°2}}$$

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Les solutions sont sous la forme :

$$\begin{cases} \theta_1(t) = A_1 \cos(\omega t + \varphi) \\ \theta_2(t) = A_2 \cos(\omega t + \varphi) \end{cases} \Rightarrow \begin{cases} \dot{\theta}_1(t) = -A_1 \omega \sin(\omega t+\varphi) \\ \dot{\theta}_2(t) = -A_2 \omega \sin(\omega t+\varphi) \end{cases} \Rightarrow \begin{cases} \ddot{\theta}_1(t) = -A_1 \omega^2 \cos(\omega t+\varphi) \\ \ddot{\theta}_2(t) = -A_2 \omega^2 \cos(\omega t + \varphi) \end{cases}$$

On remplace $\theta_1, \theta_2, \ddot{\theta}_1, \ddot{\theta}_2$ dans les eqs du mvt.

$$-mL^2 A_1 \omega^2 \cancel{\cos(\omega t+\varphi)} + \left(mgL + k\frac{L^2}{4}\right) A_1 \cancel{\cos(\omega t+\varphi)} - k\frac{L^2}{4} A_2 \cancel{\cos(\omega t+\varphi)} = 0$$

$$-mL^2 A_2 \omega^2 \cancel{\cos(\omega t+\varphi)} + \left(mgL + k\frac{L^2}{4}\right) A_2 \cancel{\cos(\omega t+\varphi)} - k\frac{L^2}{4} A_1 \cancel{\cos(\omega t+\varphi)} = 0$$

$\boxed{\text{on divise sur } \cos(\omega t+\varphi)}$

$$\begin{cases} \left(-mL^2\omega^2 + mgL + k\frac{L^2}{4}\right) A_1 - k\frac{L^2}{4} A_2 = 0 \\ \left(-mL^2\omega^2 + mgL + k\frac{L^2}{4}\right) A_2 - k\frac{L^2}{4} A_1 = 0 \end{cases}$$

Forme matricielle

$$\begin{bmatrix} -mL^2\omega^2 + mgL + k\frac{L^2}{4} & -k\frac{L^2}{4} \\ -k\frac{L^2}{4} & -mL^2\omega^2 + mgL + k\frac{L^2}{4} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

$$[B] \cdot [A] = 0$$

On calcul le déterminant, pour calculer les pulsations du système :

$$\left(-mL^2\omega^2 + mgL + k\frac{L^2}{4}\right)^2 - \left(-\frac{kL^2}{4}\right)^2 = 0$$

$$\left(-mL^2\omega^2 + mgL + k\frac{L^2}{4} + k\frac{L^2}{4}\right)\left(-mL^2\omega^2 + mgL\right) = 0$$

$$\begin{cases} -mL^2\omega^2 + mgL + \frac{kL^2}{2} = 0 \\ -mL^2\omega^2 + mgL = 0 \end{cases} \Rightarrow \boxed{\begin{aligned} \omega_1 &= \sqrt{\left(mgL + \frac{kL^2}{2}\right)/mL^2} \\ \omega_2 &= \sqrt{\frac{\cancel{m}gL}{\cancel{m}L^2}} = \sqrt{\frac{g}{l}} \end{aligned}}$$

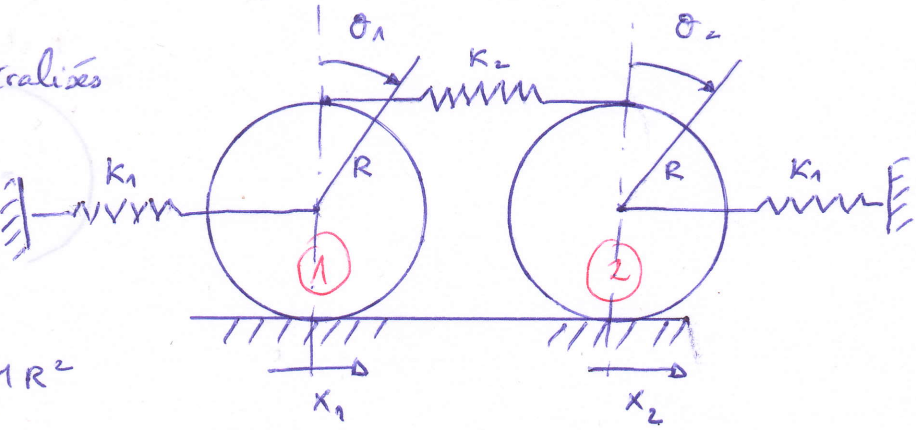
mode Harmonique

mode fondamentale

- on a $\omega_2 < \omega_1$, Donc
  - ω_2 c'est la pulsation de mode fondamentale.
  - ω_1 " " " " " Harmonique.

Exo 2

$x_1 = R.\theta_1$
$x_2 = R.\theta_2$ } Les coordonnées généralisées



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

Eq de lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$J = \frac{1}{2} M R^2$

Energie cinétique :

$$T = \underbrace{\frac{1}{2} M \dot{x}_1^2}_{\text{Ec translation disque 1}} + \underbrace{\frac{1}{2} M \dot{x}_2^2}_{\text{Ec translation disque 2}} + \underbrace{\frac{1}{2} J . \dot{\theta}_1^2}_{\text{Ec Rotation disque 1}} + \underbrace{\frac{1}{2} J \dot{\theta}_2^2}_{\text{Ec Rotation disque 2}} = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} M \dot{x}_2^2 + \frac{1}{4} M \underbrace{R^2 \dot{\theta}_1^2}_{\dot{x}_1^2} + \frac{1}{4} M \underbrace{R^2 \dot{\theta}_2^2}_{\dot{x}_2^2}$$

$$T = \frac{3}{4} M \dot{x}_1^2 + \frac{3}{4} M \dot{x}_2^2$$

Energie potentielle :

$$V = \underbrace{\frac{1}{2} K_1 x_1^2}_{\text{Ep elastique Ressort } K_1 \text{ à gauche}} + \underbrace{\frac{1}{2} K_1 x_2^2}_{\text{Ep elastique Ressort } K_1 \text{ à droite}} + \underbrace{\frac{1}{2} K_2 \Big[\underbrace{(x_2 + \overbrace{R\theta_2}^{x_2})}_{\text{déformation due à la rotation et le dep du disque 2}} - \underbrace{(x_1 + \overbrace{R\theta_1}^{x_1})}_{\text{déformation du à la Rot et le dep du disque 1}} \Big]^2}_{\text{Ep élastique du Ressort } K_2}$$

$$V = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_1 x_2^2 + \frac{1}{2} K_2 (2x_2 - 2x_1)^2$$

Lagrangien : $L = T - V = \frac{3}{4} M \dot{x}_1^2 + \frac{3}{4} M \dot{x}_2^2 - \frac{1}{2} K_1 x_1^2 - \frac{1}{2} K_1 x_2^2 - \frac{1}{2} K_2 (2x_2 - 2x_1)^2$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{3}{2} M \ddot{x}_1 \quad , \quad \frac{\partial L}{\partial x_1} = -K_1 x_1 + 2K_2 (2x_2 - 2x_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = \frac{3}{2} M \ddot{x}_1 + K_1 x_1 - 2K_2 (2x_2 - 2x_1) = 0$$

$$\frac{3}{2} M \ddot{x}_1 + K_1 x_1 - 4 K_2 x_2 + 4 K_2 x_1 = 0$$

$$\boxed{\frac{3}{2} M \ddot{x}_1 + (K_1 + 4K_2) x_1 - 4 K_2 x_2 = 0} \ldots \boxed{\text{Eq n°1}}$$

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$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{3}{2} M \ddot{x}_2 \quad , \quad \frac{\partial L}{\partial x_2} = -K_1 x_2 - 2K_2 (2x_2 - 2x_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = \frac{3}{2} M \ddot{x}_2 + K_1 x_2 + 2K_2 (2x_2 - 2x_1) = 0$$

$$\frac{3}{2} M \ddot{x}_2 + K_1 x_2 + 4 K_2 x_2 - 4 K_2 x_1 = 0$$

$$\boxed{\frac{3}{2} M \ddot{x}_2 + (K_1 + 4K_2) x_2 - 4 K_2 x_1 = 0} \quad \cdots \quad \boxed{\text{Eq n° 02}}$$

$$\Downarrow$$

Eqs du mvt
$$\begin{cases} \frac{3}{2} M \ddot{x}_1 + (K_1 + 4K_2) x_1 - 4K_2 x_2 = 0 \\ \frac{3}{2} M \ddot{x}_2 + (K_1 + 4K_2) x_2 - 4K_2 x_1 = 0 \end{cases}$$

Les solutions de ces eqs sont donné sous les formes :

$$\begin{cases} x_1(t) = A_1 \cos(\omega t + \varphi) \\ x_2(t) = A_2 \cos(\omega t + \varphi) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = -A_1 \omega \sin(\omega t + \varphi) \\ \dot{x}_2(t) = -A_2 \omega \sin(\omega t + \varphi) \end{cases} \Rightarrow \begin{cases} \ddot{x}_1(t) = -A_1 \omega^2 \cos(\omega t + \varphi) \\ \ddot{x}_2(t) = -A_2 \omega^2 \cos(\omega t + \varphi) \end{cases}$$

On remplace $x_1, x_2, \ddot{x}_1, \ddot{x}_2$ dans les eqs du mvt :

$$\begin{cases} -\frac{3}{2} M A_1 \omega^2 \cancel{\cos(\omega t + \varphi)} + (K_1 + 4K_2) A_1 \cancel{\cos(\omega t + \varphi)} - 4K_2 A_2 \cancel{\cos(\omega t + \varphi)} = 0 \\ -\frac{3}{2} M A_2 \omega^2 \cancel{\cos(\omega t + \varphi)} + (K_1 + 4K_2) A_2 \cancel{\cos(\omega t + \varphi)} - 4K_2 A_1 \cancel{\cos(\omega t + \varphi)} = 0 \end{cases}$$

$$\Downarrow$$

forme matricielle

$$\Downarrow$$

$$\begin{bmatrix} -\frac{3}{2} M\omega^2 + K_1 + 4K_2 & -4K_2 \\ -4K_2 & -\frac{3}{2} M\omega^2 + K_1 + 4K_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

$$[B] \cdot [A] = 0$$

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Le déterminant $\Delta = 0$:

$$\left(-\frac{3}{2} M\omega^2 + K_1 + 4K_2\right)^2 - (-4K_2)^2 = 0 \Rightarrow \left(-\frac{3}{2} M\omega^2 + K_1\right)\left(-\frac{3}{2} M\omega^2 + K_1 + 8K_2\right) = 0$$

$$\begin{cases} -\frac{3}{2} M\omega^2 + K_1 = 0 \Rightarrow \omega_1 = \sqrt{\frac{2K_1}{3M}} \\ -\frac{3}{2} M\omega^2 + K_1 + 8K_2 = 0 \Rightarrow \omega_2 = \sqrt{\frac{2(K_1 + 8K_2)}{3M}} \end{cases} \Rightarrow \omega_1 < \omega_2$$

→ ω_1 pulsation fondamentale

→ ω_2 pulsation Harmonique