Solution de la série TD N°2
(Les vecteurs)

Sol Ex1

1) $|\vec{V_1}| = \sqrt{(3)^2 + (4)^2 + (4)^2} = 6{,}40$

$|\vec{V_2}| = \sqrt{(2)^2 + (3)^2 + (4)^2} = 5{,}38$

$|\vec{V_3}| = \sqrt{(5)^2 + (1)^2 + (3)^2} = 5{,}91$

2) $\vec{A} = 10\vec{i} - 2\vec{j} + 3\vec{k}$ $\quad |\vec{A}| = \sqrt{(10)^2 + (2)^2 + (3)^2} = 10{,}63$

$\vec{B} = 9\vec{i} - 15\vec{j} + 15\vec{k}$ $\quad |\vec{B}| = \sqrt{(9)^2 + (15)^2 + (15)^2} = 23{,}04$

3) $\vec{C} = 8\vec{i} - 5\vec{j} + 7\vec{k}$

$\vec{u_c} = \frac{\vec{C}}{C} = \frac{8}{\sqrt{138}}\vec{i} - \frac{5}{\sqrt{138}}\vec{j} + \frac{7}{\sqrt{138}}\vec{k}$

$\vec{V_1}\cdot\vec{V_3} = x_1x_3 + y_1y_3 + z_1z_3$

$\vec{V_1}\cdot\vec{V_3} = 15 + 4 + 12$

$\vec{V_1}\cdot\vec{V_3} = 31$

$\cos\alpha = \frac{\vec{V_1}\cdot\vec{V_3}}{V_1 V_3} = \frac{31}{\sqrt{41}\cdot\sqrt{35}} = \frac{31}{37{,}88} = 0{,}82 \Rightarrow \alpha = 35{,}1°$

4) $\vec{V_2}\wedge\vec{V_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ + & - & + \\ 2 & 3 & -4 \\ 5 & -1 & 3 \end{vmatrix} = 5\vec{i} - 26\vec{j} - 17\vec{k}$

$\vec{Z} = x\vec{i} + y\vec{j} + z\vec{k}$ ou $\vec{A} + \vec{B} + \vec{Z} = \vec{0}$

$\vec{A} + \vec{B} + \vec{Z} = \begin{pmatrix} 10 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ -15 \\ 15 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 10 + 9 + x \\ -2 + 15 + y \\ 3 + 15 + z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$19 + x = 0$ $\quad x = -19$

$13 + y = 0 \Rightarrow y = -13$ $\quad$ alors $\boxed{\vec{Z} = -19\vec{i} - 13\vec{j} - 18\vec{k}}$

$18 + z = 0$ $\quad z = -18$

Sol Ex2.

$$\vec{A} = xz\,\vec{i} + (2x^2 - y)\,\vec{j} - yz^2\,\vec{k}$$

$$\Phi(x, y, z) = 3x^2 y + 2y^2 z^3$$

$$\overrightarrow{grad}(\Phi) = \vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j} + \frac{\partial \Phi}{\partial z}\vec{k}$$

$$\boxed{\vec{\nabla}\Phi = 6xy\,\vec{i} + (3x^2 + 4yz^3)\,\vec{j} + (6y^2 z^2)\,\vec{k}}$$

$$div(\vec{A}) = \vec{\nabla}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\boxed{div(\vec{A}) = z - 1 + 2yz}$$

$$\overrightarrow{rot}(\vec{A}) = \vec{\nabla} \wedge \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = z^2\vec{i} + x\vec{j} + 4x\vec{k}$$

2) Au point $(1, 0, 1)$

$$\overrightarrow{grad}(\Phi) = 3\vec{j}$$

$$div(\vec{A}) = 0$$

$$\overrightarrow{rot}(\vec{A}) = \vec{i} + \vec{j} + 4\vec{k}$$

Sol Ex 3:

1) $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

On sait que $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos(\vec{A},\vec{B}) = A\cdot B\cos(\vec{A},\vec{B})$

$$\cos(\vec{A},\vec{B}) = \frac{\vec{A}\cdot\vec{B}}{A\cdot B}$$

$$\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$$

$$A\cdot B = \sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$\cos(\vec{A},\vec{B}) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

2) $M_1(1,1,1)$ $M_2(2,2,1)$ $M_3(2,1,0)$

l'angle M_1 est entre les deux vecteurs $\overline{M_1M_2}$ et $\overline{M_1M_3}$

$$\overline{M_1M_2} = \begin{pmatrix}2\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} = \vec{i}+\vec{j} \Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_3 = 0 \end{cases}$$

$$\overline{M_1M_3} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} = \vec{i}-\vec{k} \Rightarrow \begin{cases} b_1 = 1 \\ b_2 = 0 \\ b_3 = -1 \end{cases}$$

$$\cos\phi = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \phi = \pm\frac{\pi}{3}$$

3) l'équation du plan (P) qui passe par le point $M_0(x_0, y_0, z_0) \in (P)$

On supposant qu'un autre point $M(x,y,z) \in (P)$ vérifie les condition $\overline{MM_0} \perp \vec{A} \Rightarrow \overline{MM_0} = A = 0$

$$\begin{pmatrix}x_0-x\\y_0-y\\z_0-z\end{pmatrix}\begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow 3(x-x_0) - 2(y-y_0) + (z-z_0) = 0 \ldots (2)$$

En cas particulier: M_0 est en $M_2 \Rightarrow \begin{cases} x_0 = 2 \\ y_0 = 2 \\ z_0 = 1 \end{cases}$

On remplaçant ces valeurs dans l'équation (2) on trouve

$3(2-x) - 2(2-y) + (1-z) = 0$ ou après calcule on trouve :

$$\boxed{-3x + 2y - z + 3 = 0}$$

Sol Ex4:

$\vec{A} = 3x\vec{i} + x^2\vec{j} - x^3\vec{k}$

$\vec{B} = -x\vec{i} + 4x\vec{j} + x\vec{k}$

1 - On appliquant les règles de dérivation vectorielle

$$\frac{d(\vec{A}.\vec{B})}{dx} = A\frac{d\vec{B}}{dx} + \frac{d\vec{A}}{dx}\vec{B} \quad \dots (1)$$

$$= \left(3x\vec{i} + x^2\vec{j} - x^3\vec{k}\right)\left(-\vec{i} + 4\vec{j} + \vec{k}\right) + \left(3\vec{i} + 2x\vec{j} - 3x^2\vec{k}\right)\left(-x\vec{i} + 4x\vec{j} + x\vec{k}\right)$$

$$\boxed{\frac{d(\vec{A}.\vec{B})}{dx} = -6x + 12x^2 - 4x^3} \quad \dots (2)$$

$$\frac{d(\vec{A}\wedge\vec{B})}{dx} = \frac{d\vec{A}}{dx}\wedge\vec{B} + \vec{A}\wedge\frac{d\vec{B}}{dx} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2x & -3x^2 \\ -1 & 4 & 1 \end{vmatrix} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3x & x^2 & -x^3 \\ -1 & 4 & 1 \end{vmatrix}$$

$$\boxed{\frac{d(\vec{A}\wedge\vec{B})}{dx} = \vec{i}\left(3x^2 + 16x^3\right) - \vec{j}\left(6x - 4x^3\right) + \vec{k}\left(24x + 3x^2\right)} \quad \dots (3)$$

2 - $\vec{A}.\vec{B} = -3x^2 + 4x^3 - x^4$

$$\vec{A}\wedge\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3x & x^2 & -x^3 \\ -x & 4x & x \end{vmatrix} = \vec{i}\left(x^3 + 4x^4\right) - \vec{j}\left(3x^2 - x^4\right) + \vec{k}\left(12x^2 + x^3\right)$$

Les dérivées:

$$\frac{d(\vec{A}.\vec{B})}{dx} = -6x + 12x^2 - 4x^3 \quad \dots \text{resultat}_{(2)}$$

$$\frac{d(\vec{A}\wedge\vec{B})}{dx} = \left(3x^2 + 16x^3\right)\vec{i} - \left(6x - 4x^3\right)\vec{j} + \left(24x + 3x^2\right) \quad \dots \text{resultat}_{(3)}$$