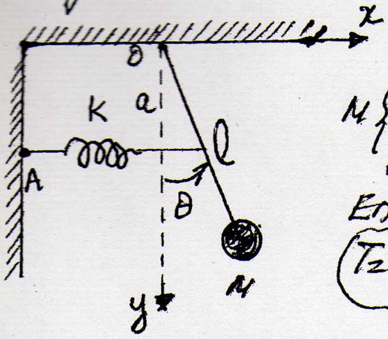
Serie supplementaire N°2

Exercice N°1

Calcul de l'equation diff. pour

Le systeme a



$M \begin{cases} x_M = l\sin\theta \\ y_M = l\cos\theta \end{cases} \Rightarrow \begin{cases} l\dot\theta\cos\theta \\ -l\dot\theta\sin\theta \end{cases}$

Energie cinetique

$T = \frac{1}{2} m l^2 \dot\theta^2$ (1pt)

Energie potentielle $U = U_m + U_K$ (1pt)

$U = -mgl\cos\theta + \frac{1}{2}K(a\sin\theta)^2$

Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2} m l^2 \dot\theta^2 + mgl\cos\theta - \frac{1}{2}K(a\sin\theta)^2$$

le Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$\frac{\partial L}{\partial\dot\theta} = m l^2\dot\theta, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = m l^2\ddot\theta$

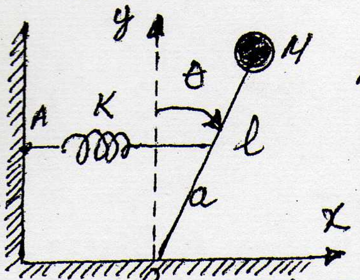
$\frac{\partial L}{\partial\theta} = -mgl\sin\theta - K(a\cos\theta)(a\sin\theta)$

$\Rightarrow m l^2\ddot\theta + mgl\sin\theta + K a^2\cos\theta\sin\theta = 0$

pour des oscillations de faibles amplitudes $\sin\theta \simeq \theta$ et $\cos\theta \simeq 1$ on aura

$$m l^2\ddot\theta + (mgl + K a^2)\theta = 0$$ (2pts)

Le systeme b



$M \begin{cases} l\sin\theta \\ l\cos\theta \end{cases} \Rightarrow \begin{cases} l\dot\theta\cos\theta \\ -l\dot\theta\sin\theta \end{cases}$

Energie cinetique

$T = \frac{1}{2} M l^2 \dot\theta^2$ (1pt)

Energie potentielle $U = U_m + U_K$

$U = mgl\cos\theta + \frac{1}{2}K(a\sin\theta)^2$ (1pt)

Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2} M l^2\dot\theta^2 - mgl\cos\theta - \frac{1}{2}K(a\sin\theta)^2$$

Le formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$\frac{\partial L}{\partial\dot\theta} = M l^2\dot\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = M l^2\ddot\theta$

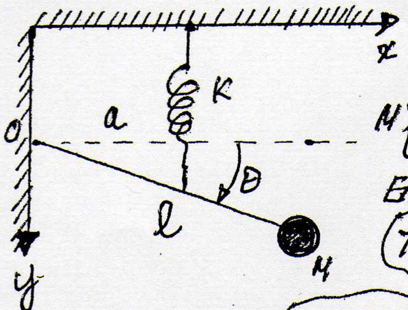
$\frac{\partial L}{\partial\theta} = mgl\sin\theta - \frac{1}{2}K(a\cos\theta)(a\sin\theta)\times 2$

$\Rightarrow M l^2\ddot\theta - mgl\sin\theta + K a^2\cos\theta\sin\theta = 0$

pour des oscillations de faibles amplitudes $\sin\theta \simeq \theta$ et $\cos\theta \simeq 1$ on aura

$$M l^2\ddot\theta + (K a^2 - mgl)\theta = 0$$ (2pts)

Le systeme c



$M \begin{cases} l\cos\theta \\ l\sin\theta \end{cases} \Rightarrow \begin{cases} -l\dot\theta\sin\theta \\ l\dot\theta\cos\theta \end{cases}$

Energie cinetique

$T = \frac{1}{2} M l^2\dot\theta^2$ (1pt)

Energie potentielle $U = \frac{1}{2}K(a\sin\theta)^2$ (1pt)

Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2} M l^2\dot\theta^2 - \frac{1}{2}K(a\sin\theta)^2$$

$\frac{\partial L}{\partial\dot\theta} = M l^2\dot\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = M l^2\ddot\theta$

$\frac{\partial L}{\partial\theta} = -K(a\cos\theta)(a\sin\theta)$

$\Rightarrow$ L'equa. diff est $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$$M l^2\ddot\theta + K a^2\theta = 0$$ (2pts)

2- Condition pour laquelle le systeme b peut osciller

Dans le cas (b), le systeme ne peut osciller que si $\frac{K}{M}\left(\frac{a}{l}\right)^2 - \frac{g}{l} > 0$ donc

$K a^2 > mgl$

sinon le mvt n'est pas oscillatoire

3- La cause pour laquelle la periode d'oscillat est independante de g dans le cas (c)

Dans le cas de la figure (c), à l'état d'équilibre les 2 forces $\vec{P} = m\vec{g}$ et $\vec{F} = -K\vec{x}_0$ se compensent entre elles $\Rightarrow$ l'energie potentielle ne peut pas contenis un terme gravitationnel $\Rightarrow$ la periode est independante de g.

Exercice 02

$m = 10g$, $E = 3{,}1\times10^{-5}J$ obeit à l'equation $x(t) = A\sin(\omega_0 t+\varphi)$ | $A = 5cm$, $\omega_0^2 = K/m$

1- Ecriture de l'equation du mvt harmonique en utilisant les valeurs numeriques des Coefficients

la phase initiale est de $\frac{\pi}{4}$ (rd)

L'energie totale $E = T+U$ | $x = A\sin(\omega_0 t+\varphi)$, $\dot{x} = A\omega_0\cos(\omega_0 t+\varphi)$

$$E = \frac{1}{2}m\dot{x}^2+\frac{1}{2}Kx^2$$

$$E = \frac{1}{2}mA^2\omega_0^2\cos^2(\omega_0 t+\varphi)+\frac{1}{2}KA^2\sin^2(\omega_0 t+\varphi)$$

sachant que $K = m\omega_0^2$ on aura

$$E = \frac{1}{2}mA^2\omega_0^2\cos^2(\omega_0 t+\varphi)+\frac{1}{2}m\omega_0^2A^2\sin^2(\omega_0 t+\varphi)$$

$$E = \frac{1}{2}mA^2\omega_0^2 \Rightarrow \frac{2E}{mA^2} = \omega_0^2 \Rightarrow \omega_0 = \sqrt{\frac{2E}{mA^2}}$$

A.N $\quad \omega_0 = \sqrt{\frac{2\times3{,}1\times10^{-5}}{10.10^{-3}\times(5\times10^{-2})^2}} = 1{,}6$ rd/s

$$\Rightarrow x(t) = 5\times10^{-2}\sin(1{,}6t+\pi/4)\ (m)$$

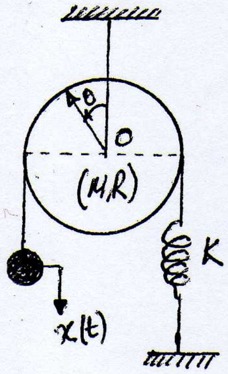
2- Calcul du temps que met le pt pour se deplacer jusqu'à sa position maximale sachant que

$$x(t) = 7\sin(0{,}5\pi t)$$

$$\omega_0 = 0{,}5\pi = \frac{2\pi}{T}\ (rd/s) \Rightarrow T = \frac{2}{0{,}5}\ (4s)$$

pour aller de l'équilibre $\rightarrow$ jusqu'à la position maximale $t = \frac{T}{4} = 1(s)$

Exercice 03



poulie homogene $J_0 = \frac{1}{2}MR^2$

Ecriture de l'equation du mvt et la pulsation propre ω_0 du système

* Energie cinetique $T = T_M+T_m$

$$T = \frac{1}{2}M\dot{X}^2+\frac{1}{2}J_0\dot{\theta}^2+\frac{1}{2}m\dot{x}^2 \qquad X = 2R\theta,\ x = 2R\theta$$

$$T = \frac{3}{4}MR^2\dot{\theta}^2+\frac{4}{2}mR^2\dot{\theta}^2 = \frac{1}{2}\left(\frac{3M}{2}+4m\right)R^2\dot{\theta}^2$$

$$\boxed{T = \frac{1}{2}\left(\frac{3M}{2}+4m\right)R^2\dot{\theta}^2}$$

* Energie potentielle $U = \frac{1}{2}K(2R\theta)^2$

* Fonction de Lagrange $L = T-U$

$$L = \frac{1}{2}\left(\frac{3M}{2}+4m\right)R^2\dot{\theta}^2-\frac{1}{2}K(2R\theta)^2$$

$$\frac{\partial L}{\partial\dot{\theta}} = \left(\frac{3M}{2}+4m\right)R^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \left(\frac{3M}{2}+4m\right)R^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta} = -K(2R)(2R\theta)$$

le Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta} = 0$

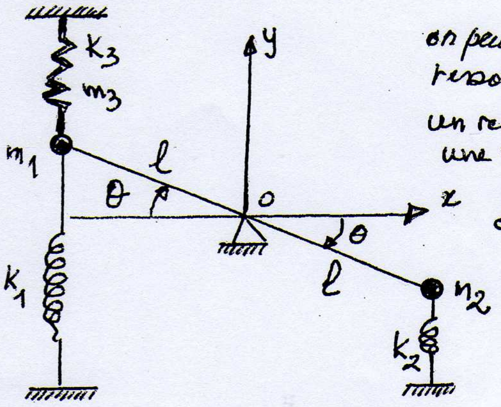
$$\left(\frac{3M}{2}+4m\right)R^2\ddot{\theta}+4KR^2\theta = 0 \quad \text{ou}$$

$$(3M+8m)\ddot{\theta}+8K\theta = 0$$

pulsation propre ω_0

$$\ddot{\theta}+\underbrace{\frac{8K}{3M+8m}}_{\omega_0^2}\theta = 0 \Rightarrow \omega_0 = \sqrt{\frac{8K}{3M+8m}}$$

Exercice 04



on peut remplacer le ressort (m_3, K_3) par un ressort K_3 avec une masse $\frac{m_3}{3}$ à sa fin

donc on aura

$m_e = m_1+\frac{m_3}{3}$

1- Ecriture de l'equation differ. du mvt

* Les Coordonnées

$$m_e\begin{cases}-l\cos\theta\\ +l\sin\theta\end{cases} \Rightarrow \begin{cases}l\dot{\theta}\sin\theta\\ l\dot{\theta}\cos\theta\end{cases}$$

$$m_2\begin{cases}+l\cos\theta\\ -l\sin\theta\end{cases} \Rightarrow \begin{cases}-l\dot{\theta}\sin\theta\\ -l\dot{\theta}\cos\theta\end{cases}$$

* Energie cinetique $T = T_{m_e}+T_{m_2}$

$$T = \frac{1}{2}\underbrace{(m_e+m_2)}_{m}l^2\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad m = m_e+m_2$$

* Energie potentielle $U = U_{K_1}+U_{K_2}+U_{K_3}$

$$U = \frac{1}{2}K_1(l\sin\theta)^2+\frac{1}{2}K_2(l\sin\theta)^2+\frac{1}{2}K_3(l\sin\theta)^2$$

$$U = \frac{1}{2}K(l\sin\theta)^2 \qquad K = K_1+K_2+K_3$$

1 Fonction de Lagrange $L = T - U = \frac{1}{2} m l^2 \dot\theta^2 - \frac{1}{2}K(l\sin\theta)^2$

: Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$$\frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = ml^2\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = -K(l\underbrace{\cos\theta}_{1})(l\underbrace{\sin\theta}_{\theta})$$

Donc l'equa. diff. s'ecrit

$$ml^2\ddot\theta + Kl^2\theta = 0 \quad \text{ou} \quad m\ddot\theta + K\theta = 0$$

2- Ecriture de la solution $\theta(t)$

$\ddot\theta + \omega_0^2\theta = 0$ Syst. libre non amorti donc la solution $\theta(t) = A\sin(\omega_0 t + \varphi)$ $\Rightarrow \omega_0^2 = \frac{K}{m}$

$$\omega_0 = \sqrt{\frac{K_1+K_2+K_3}{m_1+\frac{m_3}{3}+m_2}}$$

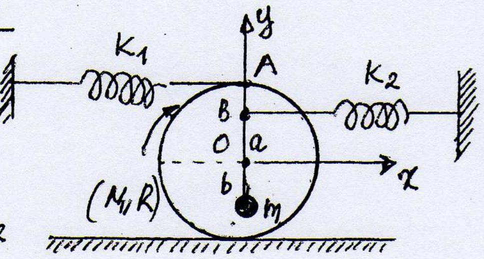
Exercice 05

$OA = R$

$Om = b$

$OB = a$

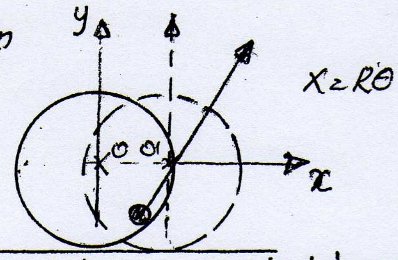
$J_0 = \frac{1}{2}MR^2$



Le disque roule sans glissement sur le plan

1- Representation du systeme en mouvement

- Calcul de la fonction de Lagrange $L = T - U$



Coordonnées

$$M\begin{cases}x = R\theta\\ 0\end{cases}$$

$$m\begin{cases}x - b\sin\theta\\ 0 - b\cos\theta\end{cases} \Rightarrow \begin{cases}R\theta - b\sin\theta\\ -b\cos\theta\end{cases} \Rightarrow \begin{cases}R\dot\theta - b\dot\theta\cos\theta\\ b\dot\theta\sin\theta\end{cases}$$

Energie cinetique $T = T_M + T_m$

$$T = \frac{1}{2}M\dot x^2 + \frac{1}{2}J_0\dot\theta^2 + \frac{1}{2}m[(R-b)^2\dot\theta^2]$$

$$T = \left(\frac{3}{4}MR^2 + \frac{1}{2}m(R-b)^2\right)\dot\theta^2 = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(R-b)^2\right]\dot\theta^2 = T$$

Energie potentielle $U = U_m + U_{K_A} + U_{K_B}$

$$U = -mgb\cos\theta + \frac{1}{2}K_1(2R\theta)^2 + \frac{1}{2}K_2(R\theta + a\theta)^2$$

Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(R-b)^2\right]\dot\theta^2 + mgb\cos\theta - \frac{1}{2}K_1(2R\theta)^2 - \frac{1}{2}K_2(R+a)^2\theta^2$$

3- Determination de l'equa. diff. du mvt

$$\frac{\partial L}{\partial\dot\theta} = \left(\frac{3}{2}MR^2 + m(R-b)^2\right)\dot\theta \Rightarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \left(\frac{3}{2}MR^2 + m(R-b)^2\right)\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = -mgb\underbrace{\sin\theta}_{\theta} - K_1(2R)(2R\theta) - K_2(R+a)^2\theta$$

$$\Rightarrow \left[\frac{3}{2}MR^2 + m(R-b)^2\right]\ddot\theta + \left[mgb + 4K_1R^2 + K_2(R+a)^2\right]\theta =$$

4- Determination de la solution $\theta(t)$

$$\ddot\theta + \underbrace{\frac{mgb + 4K_1R^2 + K_2(R+a)^2}{\frac{3}{2}MR^2 + m(R-b)^2}}_{\omega_0^2}\theta = 0$$

$$\omega_0 = \sqrt{\frac{mgb + 4K_1R^2 + K_2(R+a)^2}{\frac{3}{2}MR^2 + m(R-b)^2}}$$

$\ddot\theta + \omega_0^2\theta = 0$ Syst. libre non amorti

donc la solution $\theta(t) = A\sin(\omega_0 t + \varphi)$

sachant que $(\theta(0) = \theta_0$ et $\dot\theta(0) = 0)$

$$\dot\theta(t) = A\omega_0\cos(\omega_0 t + \varphi)$$

$$\theta(0) = A\sin\varphi = \theta_0$$

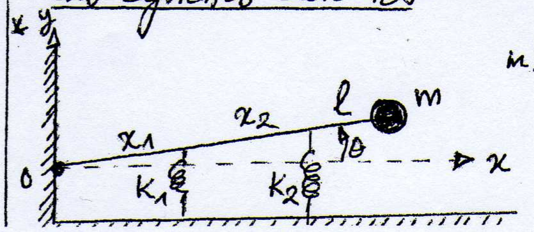
$$\dot\theta(0) = A\omega_0\cos\varphi = 0 \Rightarrow \cos\varphi = 0 \Rightarrow$$

$$\varphi = \pi/2 \Rightarrow A = \theta_0$$

$$\theta(t) = \theta_0\sin(\omega_0 t + \pi/2)$$

Exercice 06

Ecriture de l'equa. du mvt pour chacun des systemes suivants:



$$m\begin{cases}+l\cos\theta\\ +l\sin\theta\end{cases} \Rightarrow \begin{cases}-l\dot\theta\sin\theta\\ l\dot\theta\cos\theta\end{cases}$$

* Energie cinetique $\left(T = \frac{1}{2} m l^2 \dot{\theta}^2\right)$

Energie potentielle $U = U_{K_1} + U_{K_2}$

$$U = \frac{1}{2} k_1 (x_1 \sin\theta)^2 + \frac{1}{2} k_2 (x_2 \sin\theta)^2$$

$$U = \frac{1}{2} (k_1 x_1^2 + k_2 x_2^2) \sin^2\theta$$

* Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} (k_1 x_1^2 + k_2 x_2^2) \sin^2\theta$$

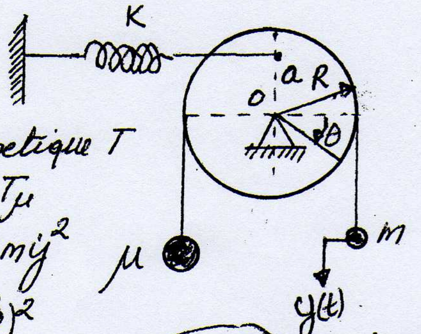
* Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -(k_1 x_1^2 + k_2 x_2^2) \underbrace{\cos\theta}_{1} \underbrace{\sin\theta}_{\theta}$$

$$\Rightarrow m l^2 \ddot{\theta} + (k_1 x_1^2 + k_2 x_2^2) \theta = 0$$

*



* Energie cinetique T

$= T_M + T_m + T_\mu$

$= \frac{1}{2} J_O \dot{\theta}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} \mu (R\dot{\theta})^2$

$$y = R\theta \Rightarrow \dot{y} = R\dot{\theta}$$

$$T = \frac{1}{2}\left(J_O + (m+\mu) R^2\right) \dot{\theta}^2$$

Energie potentielle $U = \frac{1}{2} K (a \sin\theta)^2$

Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2}\left(J_O + (m+\mu) R^2\right) \dot{\theta}^2 - \frac{1}{2} K (a \sin\theta)^2$$

Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = \left(J_O + (m+\mu) R^2\right) \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(J_O + (m+\mu) R^2\right) \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -K (a \underbrace{\cos\theta}_{1}) (a \underbrace{\sin\theta}_{\theta})$$

$$\Rightarrow \left(J_O + (m+\mu) R^2\right) \ddot{\theta} + K a^2 \theta = 0$$

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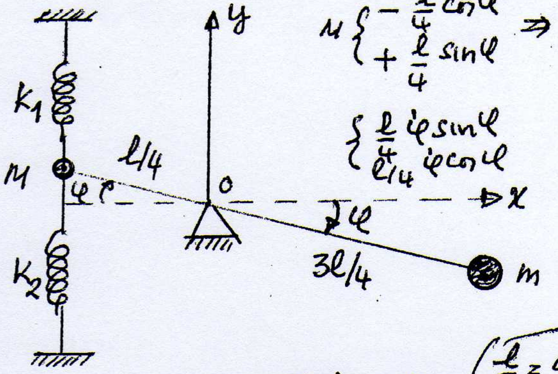

$$M \begin{cases} -\frac{l}{4} \cos\varphi \\ +\frac{l}{4} \sin\varphi \end{cases} \Rightarrow \begin{cases} \frac{l}{4} \dot{\varphi} \sin\varphi \\ \frac{l}{4} \dot{\varphi} \cos\varphi \end{cases}$$

$$m \begin{cases} +\frac{3l}{4} \cos\varphi \\ -\frac{3l}{4} \sin\varphi \end{cases} \Rightarrow \begin{cases} -\frac{3l}{4} \dot{\varphi} \sin\varphi \\ -\frac{3l}{4} \dot{\varphi} \cos\varphi \end{cases}$$

$$\frac{l}{4} = a \qquad \frac{3l}{4} = b$$

* Energie cinetique $T = T_M + T_m = \frac{1}{2} M a^2 \dot{\varphi}^2 + \frac{1}{2} m b^2 \dot{\varphi}^2$

$$T = \frac{1}{2} (M a^2 + m b^2) \dot{\varphi}^2$$

* Energie potentielle $U = U_{K_1} + U_{K_2}$

$$U = \frac{1}{2} K_1 (a \sin\varphi)^2 + \frac{1}{2} K_2 (a \sin\varphi)^2$$

$$U = \frac{1}{2} K (a \sin\varphi)^2 \quad / \quad K = K_1 + K_2$$

* Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2} (M a^2 + m b^2) \dot{\varphi}^2 - \frac{1}{2} K (a \sin\varphi)^2$$

* Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$

$$\frac{\partial L}{\partial \dot{\varphi}} = (M a^2 + m b^2) \dot{\varphi} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = (M a^2 + m b^2) \ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = -K (a \underbrace{\cos\varphi}_{1}) (a \underbrace{\sin\varphi}_{\varphi})$$

$$\Rightarrow (M a^2 + m b^2) \ddot{\varphi} + K a^2 \varphi = 0$$

$$\left(\frac{M l^2}{16} + \frac{9 m l^2}{16}\right) \ddot{\varphi} + K \frac{l^2}{16} \varphi = 0$$

$$(M + 9m) \ddot{\varphi} + K \varphi = 0$$

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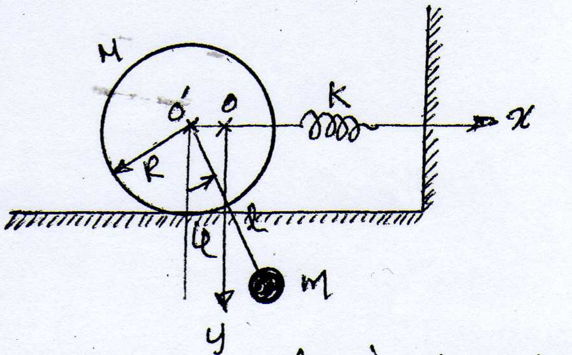


Le disque roule sans glissement sur le pla

* Coordonnées

$J_{/0} = \frac{1}{2} MR^2$

$M \begin{cases} -R\varphi \\ 0 \end{cases} \Rightarrow \begin{cases} -R\dot{\varphi} \\ 0 \end{cases}$

$m \begin{cases} -R\varphi + l\sin\varphi \\ 0 + l\cos\varphi \end{cases} \Rightarrow \begin{cases} -R\dot{\varphi} + l\dot{\varphi}\cos\varphi \\ -l\dot{\varphi}\sin\varphi \end{cases}$

* Energie cinétique $T = T_M + T_m$

$T_M = \frac{1}{2} MR^2\dot{\varphi}^2 + \frac{1}{2} J_{/0}\dot{\varphi}^2 = \frac{3}{4} MR^2\dot{\varphi}^2$

$T_m = \frac{1}{2} m(l-R)^2\dot{\varphi}^2$

$$T = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(l-R)^2\right]\dot{\varphi}^2$$

* Energie potentielle $U = U_m + U_K$

$$U = -mgl\cos\varphi + \frac{1}{2}KR^2\varphi^2$$

* Fonction de Lagrange $L = T - U$

$L = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(l-R)^2\right]\dot{\varphi}^2 + mgl\cos\varphi - \frac{1}{2}KR^2\varphi^2$

Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$

$\frac{\partial L}{\partial \dot{\varphi}} = \left[\frac{3}{2}MR^2 + m(l-R)^2\right]\dot{\varphi}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \left[\frac{3}{2}MR^2 + m(l-R)^2\right]\ddot{\varphi}$

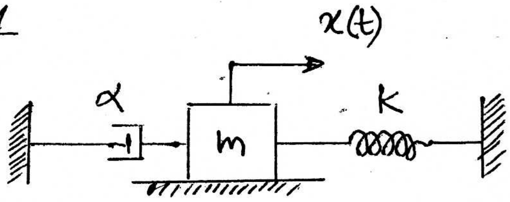
$\frac{\partial L}{\partial \varphi} = -mgl\underbrace{\sin\varphi}_{\varphi} - KR^2\varphi$

Donc l'équa. diff. s'écrit

$$\left[\frac{3}{2}MR^2 + m(l-R)^2\right]\ddot{\varphi} + (mgl + KR^2)\varphi = 0$$

Serie supplementaire N°3

Exercice N°1



1 Determination de l'equa. diff. en fonction de δ et ω_0

- Energie cinetique $T = \frac{1}{2} m\dot{x}^2$ (0,5)
. Energie potentielle $U = \frac{1}{2} K x^2$ (0,5)

Fonction de dissipation $D = \frac{1}{2}\alpha \dot{x}^2$ (0,5)

Fonction de Lagrange $L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2}Kx^2$

Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = -\frac{\partial D}{\partial \dot{x}}$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

$\frac{\partial L}{\partial x} = -Kx$, $\frac{\partial D}{\partial \dot{x}} = \alpha\dot{x}$ $\Rightarrow m\ddot{x} + Kx = -\alpha\dot{x}$

L'equa. diff. s'ecrit $m\ddot{x} + \alpha\dot{x} + Kx = 0$ (0,5)

$\Rightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{K}{m}x = 0 \Rightarrow \ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ (0,5)

tel que $2\delta = \frac{\alpha}{m}$ et $\omega_0^2 = \frac{K}{m}$

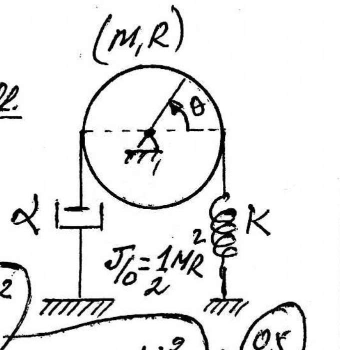
déduction de $\omega_a = \sqrt{\omega_0^2 - \delta^2} = \sqrt{\frac{K}{m} - \frac{\alpha^2}{4m^2}} = \omega_a$ (0,5)

Ecriture de la solution de l'equa. diff. pour $\delta < \omega_0$

$x(t) = C e^{-\delta t}\sin(\omega_a t + \varphi)$ (1pt)

Exercice N°2

Determination de l'equa. diff. en fonction de δ et ω_0



Energie cinetique (0,5)
$T = \frac{1}{2}J_{/0}\dot{\theta}^2$

Energie potentielle $U = \frac{1}{2}K(R\theta)^2$ (0,5)

Fonction de dissipation $D = \frac{1}{2}\alpha(R\dot{\theta})^2$ (0,5)

Fonction de Lagrange $L = T - U = \frac{1}{2}J_{/0}\dot{\theta}^2 - \frac{1}{2}K(R\theta)^2$

Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\partial D}{\partial \dot{\theta}}$

$\frac{\partial L}{\partial \dot{\theta}} = J_{/0}\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J_{/0}\ddot{\theta}$

$\frac{\partial L}{\partial \theta} = -KR^2\theta$, $\frac{\partial D}{\partial \dot{\theta}} = \alpha R^2\dot{\theta}$

$\Rightarrow J_{/0}\ddot{\theta} + KR^2\theta = -\alpha R^2\dot{\theta} \Rightarrow$ (0,5)

$\frac{1}{2}MR^2\ddot{\theta} + \alpha R^2\dot{\theta} + KR^2\theta = 0 \Rightarrow$

$M\ddot{\theta} + 2\alpha\dot{\theta} + 2K\theta = 0$ ou

$\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = 0$ (0,5) tel que $\begin{cases}\delta = \frac{\alpha}{M} \\ \omega_0^2 = \frac{2K}{M}\end{cases}$ (0,5)

* Solution de l'equa. diff lorsque $\delta < \omega_0$

$\theta(t) = Ce^{-\delta t}\cos(\omega_a t + \varphi)$ / $\omega_a = \sqrt{\frac{2K}{M} - \frac{\alpha^2}{M^2}}$

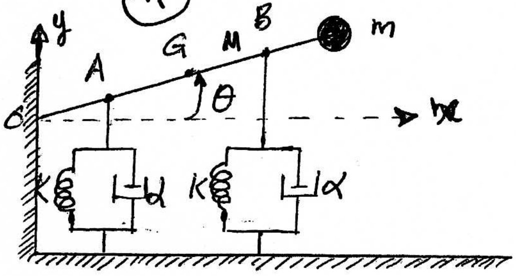
(1pt)

Exercice 03

$J_{/G} = \frac{1}{12}ML^2$

$OA = L_1$

$OB = L_2$



1- Ecriture de l'equation diff. du mvt

* Coordonnées

$M\begin{cases} l\cos\theta \\ l\sin\theta \end{cases} \Rightarrow \begin{cases} -L\dot{\theta}\sin\theta \\ L\dot{\theta}\cos\theta \end{cases}$

* Energie cinetique $T = T_M + T_m$

$T = \frac{1}{2}J_{/0}\dot{\theta}^2 + \frac{1}{2}mL^2\dot{\theta}^2$ / $J_{/0} = J_{/G} + M\left(\frac{L}{2}\right)^2$

$J_{/0} = \frac{1}{12}ML^2 + M\frac{L^2}{4}\cdot\frac{3}{3} = \frac{1}{3}ML^2$

$T = \frac{1}{2}\left(\frac{M}{3} + m\right)L^2\dot{\theta}^2$ (1pt)

* Energie potentielle $U = U_{K_A} + U_{K_B}$

$U = \frac{1}{2}K(L_1\sin\theta)^2 + \frac{1}{2}K(L_2\sin\theta)^2$ (1pt)

* Fonction de dissipation $D = D_{\alpha_A} + D_{\alpha_B}$

$D = \frac{1}{2}\alpha(L_1\dot{\theta}\cos\theta)^2 + \frac{1}{2}\alpha(L_2\dot{\theta}\cos\theta)^2$ (1pt)

* La Fonction de Lagrange $L = T - U$

$L = \frac{1}{2}\left(\frac{M}{3} + m\right)L^2\dot{\theta}^2 - \frac{1}{2}K(L_1\sin\theta)^2 - \frac{1}{2}K(L_2\sin\theta)^2$

* Le Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\partial D}{\partial \dot{\theta}}$

$\frac{\partial L}{\partial \dot{\theta}} = \left(\frac{M}{3} + m\right)L^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\frac{M}{3} + m\right)L^2\ddot{\theta}$

$\frac{\partial L}{\partial \theta} = -K(L_1\cos\theta)(L_1\sin\theta) - K(L_2\cos\theta)(L_2\sin\theta)$

$\frac{\partial D}{\partial \dot{\theta}} = \alpha(L_1\cos\theta)(L_1\dot{\theta}\cos\theta) + \alpha(L_2\cos\theta)(L_2\dot{\theta}\cos\theta)$

L'équa. diff. s'écrit comme suit pour des oscillations de faible amplitude $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$

$$\left(\frac{M}{3}+m\right)L^2\ddot{\theta}+(L_1^2+L_2^2)K\theta=-\alpha(L_1^2+L_2^2)\dot{\theta}$$

$$\left(\frac{M}{3}+m\right)L^2\ddot{\theta}+\alpha(L_1^2+L_2^2)\dot{\theta}+K(L_1^2+L_2^2)\theta=0$$ 1pt

Déduction de ω_0 et δ

On peut écrire l'équat. diff. comme

$$\ddot{\theta}+\underbrace{\frac{\alpha(L_1^2+L_2^2)}{\left(\frac{M}{3}+m\right)L^2}}_{2\delta}\dot{\theta}+\underbrace{\frac{K(L_1^2+L_2^2)}{\left(\frac{M}{3}+m\right)L^2}}_{\omega_0^2}\theta=0$$

$$2\delta=\frac{\alpha(L_1^2+L_2^2)}{\left(\frac{M}{3}+m\right)L^2} \Rightarrow \delta=\frac{\alpha(L_1^2+L_2^2)}{2\left(\frac{M}{3}+m\right)L^2}$$

$$\omega_0^2=\frac{K(L_1^2+L_2^2)}{\left(\frac{M}{3}+m\right)L^2} \Rightarrow \omega_0=\sqrt{\frac{K(L_1^2+L_2^2)}{\left(\frac{M}{3}+m\right)L^2}}$$

Ecriture de l'équa. du mvt dans le cas $\delta < \omega_0$

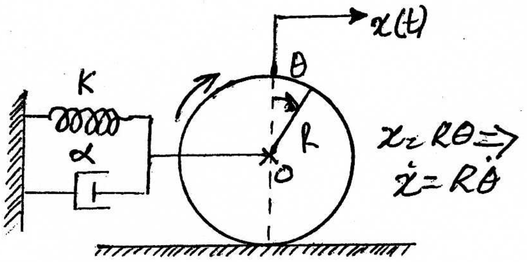
La solution est de la forme

$$\theta(t)=Ce^{-\delta t}\sin(\omega_a t+\varphi)$$

Exercice N°4

disque roule sans glisser

$J_0=\frac{1}{2}MR^2$



$x=R\theta \Rightarrow \dot{x}=R\dot{\theta}$

* Détermination de l'équa. diff. du mvt

$$T=\frac{1}{2}J_0\dot{\theta}^2+\frac{1}{2}M\dot{x}^2=\frac{1}{2}\cdot\frac{1}{2}MR^2\dot{\theta}^2+\frac{1}{2}MR^2\dot{\theta}^2=\frac{3}{4}MR^2\dot{\theta}^2$$

$$U=\frac{1}{2}K(R\theta)^2, \quad D=\frac{1}{2}\alpha(R\dot{\theta})^2$$ 1,5

$$L=T-U=\frac{3}{4}MR^2\dot{\theta}^2-\frac{1}{2}K(R\theta)^2$$

$$\frac{\partial L}{\partial\dot{\theta}}=\frac{3}{2}MR^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\frac{3}{2}MR^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=-KR^2\theta, \quad \frac{\partial D}{\partial\dot{\theta}}=\alpha R^2\dot{\theta}$$

Donc le formalisme de Lagrange s'écrit

$$\frac{3}{2}MR^2\ddot{\theta}+KR^2\theta=-\alpha R^2\dot{\theta} \Rightarrow$$

$$3M\ddot{\theta}+2\alpha\dot{\theta}+2K\theta=0$$ 1pt

Déduction de δ et ω_0

$$\ddot{\theta}+\underbrace{\frac{2\alpha}{3M}}_{2\delta}\dot{\theta}+\underbrace{\frac{2K}{3M}}_{\omega_0^2}\theta=0$$

$$2\delta=\frac{2\alpha}{3M} \Rightarrow \delta=\frac{\alpha}{3M}, \quad \omega_0^2=\frac{2K}{3M} \Rightarrow \omega_0=\sqrt{\frac{2K}{3M}}$$

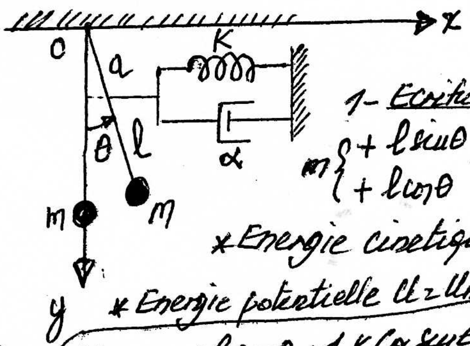
Equation diff $\ddot{\theta}+2\delta\dot{\theta}+\omega_0^2\theta=0$

$$\omega_a=\sqrt{\omega_0^2-\delta^2}=\sqrt{\frac{2K}{3M}-\frac{\alpha^2}{9M^2}}=\omega_a$$ 0,5

* Trouvons ~~l'équation~~ du mvt qd $\delta=\omega_0$

$$\theta(t)=(C_1+C_2t)e^{-\delta t}$$ 1pt

Exercice N°5



1- Ecriture de l'équa. diff

$$m\begin{cases}+l\sin\theta\\+l\cos\theta\end{cases} \Rightarrow \begin{cases}l\dot{\theta}\cos\theta\\-l\dot{\theta}\sin\theta\end{cases}$$

* Energie cinétique $T=\frac{1}{2}ml^2\dot{\theta}^2$ 0,5
* Energie potentielle $U=U_m+U_K$

$$U=-mgl\cos\theta+\frac{1}{2}K(a\sin\theta)^2$$ 1pt

* Fonction de dissipation $D=\frac{1}{2}\alpha(a\dot{\theta}\cos\theta)^2$ 0,5
* Fonction de Lagrange

$$L=T-U=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta-\frac{1}{2}K(a\sin\theta)^2$$

Le formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=-\frac{\partial D}{\partial\dot{\theta}}$

$$\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=-mgl\underbrace{\sin\theta}_{\theta}-K(a\underbrace{\cos\theta}_{1})(a\underbrace{\sin\theta}_{\theta})$$

$$\frac{\partial D}{\partial\dot{\theta}}=\alpha(a\underbrace{\cos\theta}_{1})(a\dot{\theta}\underbrace{\cos\theta}_{1})$$

$$\Rightarrow ml^2\ddot{\theta}+(mgl+Ka^2)\theta=-\alpha a^2\dot{\theta}$$

Equa. diff. $\Rightarrow ml^2\ddot{\theta}+\alpha a^2\dot{\theta}+(mgl+Ka^2)\theta=0$ 1pt

* Détermination de la pulsation propre ω_a

On peut écrire l'équa. diff. comme suit

$$\ddot{\theta}+\underbrace{\frac{\alpha a^2}{ml^2}}_{2\delta}\dot{\theta}+\underbrace{\frac{mgl+Ka^2}{ml^2}}_{\omega_0^2}\theta=0$$

$$\delta=\frac{\alpha a^2}{2ml^2}, \quad \omega_0=\sqrt{\frac{mgl+Ka^2}{ml^2}}$$

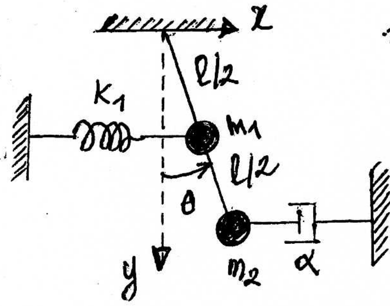
a pulsation propre $\omega_a = \sqrt{\omega_0^2 - \delta^2}$

$$\omega_a = \sqrt{\frac{mgl+Ka^2}{ml^2} - \frac{\alpha^2 a^4}{4m^2 l^4}} \to 0,5$$

* Ecriture de la solution du mvt qd $\delta < \omega_0$

$$\theta(t) = C e^{-\delta t}\sin(\omega_a t + \varphi) \to 0,5$$

Exercice 06



1- Ecriture de l'equa. diff du mvt

$m_1 \begin{cases} +\frac{l}{2}\sin\theta \\ +\frac{l}{2}\cos\theta \end{cases}$ $\quad m_1 \begin{cases} \frac{l}{2}\dot\theta\cos\theta \\ -\frac{l}{2}\dot\theta\sin\theta \end{cases}$

$$m_2 \begin{cases} +l\sin\theta \\ +l\cos\theta \end{cases} \Rightarrow \begin{cases} l\dot\theta\cos\theta \\ -l\dot\theta\sin\theta \end{cases}$$

$\Rightarrow$ Energie cinetique $T = T_{m_1} + T_{m_2} = \frac{1}{2} m_1 \frac{l^2}{4}\dot\theta^2 + \frac{1}{2} m_2 l^2 \dot\theta^2$

$$T = \frac{1}{2}\left(\frac{m_1}{4} + m_2\right) l^2 \dot\theta^2 \to 0,5$$

Energie potentielle $U = U_{m_1} + U_{m_2} + U_K = -m_1 g \frac{l}{2}\cos\theta - m_2 g l\cos\theta + \frac{1}{2}K\left(\frac{l}{2}\sin\theta\right)^2$

$$U = -\left(\frac{m_1}{2} + m_2\right) g l \cos\theta + \frac{1}{2}K\left(\frac{l}{2}\sin\theta\right)^2 \to 1pt$$

- Fonction de Dissipation $D = \frac{1}{2}\alpha\left(l\dot\theta\cos\theta\right)^2 \to 0,5$

Fonction de Lagrange $L = T - U$

$$L = \frac{1}{2}\left(\frac{m_1}{4} + m_2\right) l^2\dot\theta^2 + \left(\frac{m_1}{2} + m_2\right) g l\cos\theta - \frac{1}{2}K\left(\frac{l}{2}\sin\theta\right)^2$$

Formalisme de Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = -\frac{\partial D}{\partial\dot\theta}$

$$\frac{\partial L}{\partial\dot\theta} = \left(\frac{m_1}{4} + m_2\right) l^2\dot\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \left(\frac{m_1}{4} + m_2\right) l^2\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = -\left(\frac{m_1}{2} + m_2\right) g l \underbrace{\sin\theta}_{\theta} - K\Big(\frac{l}{2}\underbrace{\cos\theta}_{1}\Big)\Big(\frac{l}{2}\underbrace{\sin\theta}_{\theta}\Big)$$

$$\frac{\partial D}{\partial\dot\theta} = \alpha\Big(l\underbrace{\cos\theta}_{1}\Big)\Big(l\dot\theta\underbrace{\cos\theta}_{1}\Big)$$

$$\left(\frac{m_1}{4} + m_2\right) l^2\ddot\theta + \left[\left(\frac{m_1}{2} + m_2\right) g l + \frac{K l^2}{4}\right]\theta = -\alpha l^2\dot\theta$$

$$\left(\frac{m_1}{4} + m_2\right) l^2\ddot\theta + \alpha l^2\dot\theta + \left[\left(\frac{m_1}{2} + m_2\right) g l + \frac{K l^2}{4}\right]\theta = 0$$

on donne $\frac{m_1}{4} = m_2 = m$ et $\frac{K_1}{4} = K$

$$\Rightarrow 2ml^2\ddot\theta + \alpha l^2\dot\theta + \left(3mgl + Kl^2\right)\theta = 0 \quad 1pt$$

* Determination la pulsation propre ω_a

On peut écrire l'equa. diff. comme

$$\ddot\theta + \underbrace{\frac{\alpha}{2m}}_{2\delta}\dot\theta + \underbrace{\left(\frac{3}{2}\frac{g}{l} + \frac{K}{2m}\right)}_{\omega_0^2}\theta = 0$$

$$2\delta = \frac{\alpha}{2m} \Rightarrow \delta = \frac{\alpha}{4m}, \quad \omega_0^2 = \frac{3g}{2l} + \frac{K}{2m}$$

$$\omega_a = \sqrt{\omega_0^2 - \delta^2} = \sqrt{\frac{3}{2}\frac{g}{l} + \frac{K}{2m} - \frac{\alpha^2}{16m^2}} \quad 0,5$$

* Determination de la solution du mvt qd $\delta < \omega_0$ pour les Conditions initiales suivante

$$\theta(0) = \theta_0 \text{ et } \dot\theta(0) = 0$$

La solution est de la forme

$$\begin{cases} \theta(t) = e^{-\delta t}\left(C_1\sin\omega_a t + C_2\cos\omega_a t\right) \quad 0,5 \\ \dot\theta(t) = -\delta e^{-\delta t}\left(C_1\sin\omega_a t + C_2\cos\omega_a t\right) + e^{-\delta t}\left(C_1\omega_a\cos\omega_a t - C_2\omega_a\sin\omega_a t\right) \end{cases}$$

$$\theta(0) = C_2 = \theta_0$$

$$\dot\theta(0) = -\delta C_2 + C_1\omega_a = 0 \Rightarrow C_1 = \frac{\delta}{\omega_a}C_2$$

$$C_1 = \frac{\delta}{\omega_a}\theta_0$$

$$\Rightarrow \theta(t) = e^{-\delta t}\left(\frac{\delta}{\omega_a}\theta_0\sin\omega_a t + \theta_0\cos\omega_a t\right)$$

$$\Rightarrow \theta(t) = \theta_0 e^{-\delta t}\left(\frac{\delta}{\omega_a}\sin\omega_a t + \cos\omega_a t\right)$$